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Weierstrass–Stone Theorems for Set-Valued Mappings

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INTRODUCTION

Let X be a locally compact Hausdorff space and $(E, \|\cdot\|)$ a normed space over $\mathbb{K}$ ($= \mathbb{R}$ or $\mathbb{C}$). If W is a vector subspace of $\mathscr{C}_0(X; E)$, the space of all continuous functions $f: X \to E$ that vanish at infinity, and φ is a mapping from X into the non-empty subsets of E, we are interested in finding necessary and sufficient conditions under which, for every $\varepsilon > 0$, there is some $g \in W$ such that $g(x) \in \varphi(x) + \{t \in E; \|t\| < \varepsilon\}$ for all $x \in X$; that is, when, for every $\varepsilon > 0$, there is an ε-approximate W-selection for φ. More generally, we shall be interested in establishing a "localization formula" for the distance of φ from W:

$$\operatorname{dist}(\varphi; W) = \inf_{g \in W} \sup_{x \in X} \sup_{y \in \varphi(x)} \| y - g(x)\|.$$

By this we mean the following: suppose W is module over a subalgebra A of $\mathscr{C}_b(X; \mathbb{K})$, the algebra of all bounded continuous $\mathbb{K}$-valued functions on X. Let Δ be the equivalence relation on X defined by A, and for each $x \in X$, let $\Delta(x)$ be the equivalence class of x modulo Δ. Under this circumstance, when can we write

$$\operatorname{dist}(\varphi; W) = \sup_{x \in X} \operatorname{dist}(\varphi | \Delta(x); W | \Delta(x))?$$

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In the language of Buck [2], when such a formula holds we say that a strong version of the Stone–Weierstrass theorem is valid. We show in Section 1 that this is true when φ is upper semicontinuous and vanishes at infinity with respect to W. (See Theorem 1.5.)

Our interest in set-valued mappings comes in part from the consideration of best simultaneous approximation: given a bounded set $F \subset \mathscr{C}_0(X; E)$ and a non-empty subset $W \subset \mathscr{C}_0(X; E)$, the *relative Chebyshev radius* of F (with respect to W) is the number

$$\operatorname{rad}_W(F) = \inf_{g \in W} \sup_{f \in F} \| g - f \|.$$

If for every $x \in X$, $\varphi(x) = \{f(x); f \in F\}$, then

$$\operatorname{dist}(\varphi; W) = \operatorname{rad}_W(F).$$

When F is totally bounded, the mapping φ is upper semicontinuous and vanishes at infinity with respect to any W, and the localization formula of $\operatorname{dist}(\varphi; W)$ yields the analogous result for the Chebyshev radius

$$\operatorname{rad}_W(F) = \sup_{x \in X} \operatorname{rad}_{W|\Delta(x)} (F|\Delta(x)).$$

(See Theorem 1.11.)

In Section 2, we apply the results of Section 1 to the case of the so-called *Weierstrass–Stone subspaces* $W = \pi^*(\mathscr{C}_0(Y; E))$ and present a formula for $\operatorname{dist}(f; W)$, where $f \in \mathscr{C}_0(X; E)$, in terms of the Chebyshev radius of $f(\pi^{-1}(y))$, which generalizes a result of Olech [4]. (See Theorem 2.2.) Even more generally we consider the case of $F \subset \mathscr{C}_0(X; E)$ a totally bounded subset and present a formula for $\operatorname{rad}_{\pi^*(\mathscr{C}_0(Y;E))}(F)$ in terms of the Chebyshev radius of $F(\pi^{-1}(y)) = \bigcup \{f(\pi^{-1}(y)); f \subset F\}$. (See Theorem 2.4.)

In Section 3, we deal with the problem of finding *weighted* approximate W-selections of φ. We solve this problem in the case $\varphi(x)$ is convex, for each $x \in X$, and φ is lower semicontinuous and vanishes at infinity with respect to $W \subset \mathscr{C}V_\infty(X; E)$, where V is the set of weights on X under consideration (see Theorem 3.5).

The main tool, in Sections 1 and 3, is a result on partitions of unity by means of functions on a closed subalgebra of $\mathscr{C}_b(X; \mathbb{K})$ due to Nachbin (see [3, Lemma 1]). In fact, we show that the reasoning in the proof of the so-called bounded case of the weighted approximation problem found in Nachbin [3] carries over to the case of set-valued mappings.

Let us explain some notation and terminology. If X and E are topological spaces, $\mathscr{C}(X; E)$ denotes the set of all continuous functions $f: X \to E$. If φ is a map from X into the non-empty subsets of E, we call such a map a *carrier* of X into E. If Δ is any equivalence relation on X, and $x \in X$, we write $\Delta(x)$ for the Δ-equivalence class containing x; that is, $\Delta(x) = \{y \in X;\ (x, y) \in \Delta\}$. If Δ is the equivalence relation determined by $A \subset \mathscr{C}(X; E)$, $\Delta(x) = \{y \in X;\ a(x) = a(y)$ for all $a \in A\}$, and each $\Delta(x)$ is a closed subset of X. If $Y \subset X$ is any non-empty subset, and $f\colon X \to S$, is any mapping, where S is a non-empty set, we denote by $f|Y$ the mapping $y \in Y \to f(y)$. If F is any family of mappings $f\colon X \to S$, we denote by $F|Y$ the set $\{f|Y;\ f \in F\}$.

1. A Strong Weierstrass–Stone Theorem for Upper Semicontinuous Carriers

Throughout Sections 1 and 2, X stands for a locally compact Hausdorff space and E stands for a normed space over $\mathbb{K}$, where $\mathbb{K} = \mathbb{R}$ of $\mathbb{K} = \mathbb{C}$. The vector subspace of $\mathscr{C}(X; E)$ consisting of all those $f \in \mathscr{C}(X; E)$ which vanish at infinity will be denoted by $\mathscr{C}_0(X; E)$. The space $\mathscr{C}_0(X; E)$ is normed by the sup–norm

$$f \to \|f\| = \sup\{\|f(x)\|;\ x \in X\}.$$

When X is compact, $\mathscr{C}_0(X; E) = \mathscr{C}(X; E)$.

If $A \subset \mathscr{C}_b(X; \mathbb{K})$ is a self-adjoint subalgebra, then for any A-module $W \subset \mathscr{C}_0(X; E)$ one has the following "strong" formulation of the Weierstrass–Stone theorem. For any $f \in \mathscr{C}_0(X; E)$ let

$$\operatorname{dist}(f; W) = \inf\{\|f - g\|;\ g \in W\}.$$

Then

$$\operatorname{dist}(f; W) = \sup_{x \in X} \operatorname{dist}(f|\Delta(x);\ W|\Delta(x)),$$

where $\Delta(x) = \{y \in X;\ a(x) = a(y)$ for all $a \in A\}$. (See Theorem 6.1 of Prolla [5].) Our aim in this section is to generalize this formula for set-valued mappings. Let φ be a carrier from X into E. We define the *distance of φ from function* $g \in \mathscr{C}_0(X; E)$ to be

$$\operatorname{dist}(\varphi; g) = \sup_{x \in X} \{\sup_{y \in \varphi(x)} \|y - g(x)\|\}$$

and the *distance of φ from a subset* $W \subset \mathscr{C}_0(X; E)$ to be

$$\operatorname{dist}(\varphi;\ W) = \inf\{\operatorname{dist}(\varphi; g);\ g \in W\}.$$

For any carrier φ one obviously has

$$\sup_{x \in X} \operatorname{dist}(\varphi|\Delta(x);\ W|\Delta(x)) \leqslant \operatorname{dist}(\varphi;\ W).$$

DEFINITION 1.1. A carrier φ of X into E is said to be *upper semicontinuous* (u.s.c.) *with respect to* $W \subset \mathscr{C}_0(X; E)$, if given $w \in W$ and $r > 0$, for each $x \in X$ such that $\varphi(x) \subset B(w(x); r)$ and each $\varepsilon > 0$, there is a neighborhood U of x such that $\varphi(y) \subset B(w(y); r + \varepsilon)$ for all $y \in U$. (If $v \in E$ and $s > 0$, we denote by $B(v; s)$ the set $\{u \in E; \|u - v\| < s\}$.)

In particular, φ is u.s.c. with respect to $W \subset \mathscr{C}_0(X; E)$ if, given $w \in W$ and $r > 0$, the set $\{x \in X; \varphi(x) \subset B(w(x); r)\}$ is open.

EXAMPLE 1.2. If $f \in \mathscr{C}_0(X; E)$, then $\varphi(x) = \{f(x)\}$, $x \in X$, is upper semicontinuous with respect to any $W \subset \mathscr{C}_0(X; E)$. Indeed, for each $w \in W$ and $r > 0$, the set $\{x \in X; \varphi(x) \subset B(w(x); r)\} = \{x \in X; \|f(x) - w(x)\| < r\}$ is open.

EXAMPLE 1.3. Let $F \subset \mathscr{C}_0(X; E)$ be an equicontinuous subset. Define a carrier φ from X into E by setting

$$\varphi(x) = \{f(x); f \in F\}$$

for all $x \in X$. Then φ is u.s.c. with respect to any $W \subset \mathscr{C}_0(X; E)$. Indeed, let $w \in W$, $r > 0$ and $x \in X$ such that $\varphi(x) \subset B(w(x); r)$ be given. Let $\varepsilon > 0$ be given. By equicontinuity, there is a neighborhood U of x such that $\|f(t) - w(t) - (f(x) - w(x))\| < \varepsilon$ for all $t \in U$ and $f \in F$. Hence $y \in U$ implies

$$\varphi(y) \subset B(w(y); r + \varepsilon).$$

In particular, if $F \subset \mathscr{C}_0(X; E)$ is totally bounded, then the carrier φ defined above is u.s.c. with respect to any $W \subset \mathscr{C}_0(X; E)$.

DEFINITION 1.4. Let φ be a carrier of X into E and let $W \subset \mathscr{C}_0(X; E)$. We say that φ *vanishes at infinity with respect to* W, if for each $w \in W$ and $\varepsilon > 0$ the set

$$\{x \in X; \varphi(x) \cap (E \backslash B(w(x); \varepsilon)) \neq \varnothing\}$$

is relatively compact.

THEOREM 1.5. *Let $A \subset \mathscr{C}_b(X; \mathbb{K})$ be a self-adjoint subalgebra and let $W \subset \mathscr{C}_0(X; E)$ be an A-module. For any carrier φ of X into E which is upper semicontinuous and vanishes at infinity with respect to W, we have*

$$\operatorname{dist}(\varphi; W) = \sup_{x \in X} \operatorname{dist}(\varphi | \Delta(x);\ W | \Delta(x)).$$

Proof. Let $\lambda = \sup\{\operatorname{dist}(\varphi | \Delta(x);\ W | \Delta(x));\ x \in X\}$. We may assume that A contains the constants. Let $0 < \varepsilon$. For each $x \in X$, there exists $g_x \in W$

such that $\operatorname{dist}(\varphi|\Delta(x);\ g_x|\Delta(x)) < \lambda + \varepsilon/4$. This means that $\|t - g_x(y)\| < \lambda + \varepsilon/4$ for all $t \in \varphi(y)$ and $y \in \Delta(x)$. Since φ is upper semicontinuous with respect to W, there is an open neighborhood U_x of x such that

$$\|t - g_x(x')\| < \lambda + \varepsilon/2 \qquad \text{for all } t \in \varphi(x'),\ x' \in U_x.$$

Clearly, $U_x \supset \Delta(x)$.

Since φ vanishes at infinity with respect to W, the closure K_x of

$$S_x = \{y \in X;\ \varphi(y) \cap (E \backslash B(w(y); \lambda + \varepsilon/2)) \neq \varnothing\}$$

is compact. We claim that $\Delta(x) \cap K_x = \varnothing$. Indeed, assume $z \in \Delta(x) \cap K_x$. Since $\Delta(x) \subset U_x$ and K_x is the closure of S_x, there is some $y \in U_x \cap S_x$. But $\varphi(y) \subset B(w(y);\ \lambda + \varepsilon/2)$ for all $y \in U_x$ and so y cannot be in S_x. By [3, Lemma 1] there exists a finite set $\{x_1, x_2, \ldots, x_n\} \subset X$ and for each $1 \leqslant i \leqslant n$, there is $\varphi_i \in \bar{A}$ such that $\varphi_i \geqslant 0$, $\varphi_i | K_{x_i} = 0$ and $\sum_{i=1}^n \varphi_i = 1$ on X. Let $g = \sum_{i=1}^n \varphi_i g_{x_i}$. Then for each $x \in X$ and $t \in \varphi(x)$ we have $\|t - g(x)\| \leqslant \lambda + \varepsilon/2$. In fact, either $x \in K_{x_i}$ and then $\varphi_i(x) = 0$; or else $x \in U_{x_i}$ and then $\|t - g_{x_i}(x)\| < \lambda + \varepsilon/2$. Hence

$$\begin{aligned}\|t - g(x)\| &= \left\| \sum_{i=1}^n \varphi_i(x)(t - g_{x_i}(x)) \right\| \\ &\leqslant \sum_{i=1}^n \varphi_i(x) \|t - g_{x_i}(x)\| \\ &\leqslant (\lambda + \varepsilon/2) \sum_{i=1}^n \varphi_i(x) = \lambda + \varepsilon/2.\end{aligned}$$

Therefore $\operatorname{dist}(\varphi; g) \leqslant \lambda + \varepsilon/2$. Choose $\delta > 0$ so that $\delta \sum_{i=1}^n \|g_{x_i}(x)\| < \varepsilon/2$ for all $x \in X$; and for each $1 \leqslant i \leqslant n$ choose $a_i \in A$ such that $|a_i(x) - \varphi_i(x)| < \delta$ for all $x \in X$. Now $h = \sum_{i=1}^n a_i g_{x_i}$ belongs to W and $\operatorname{dist}(\varphi; h) \leqslant \lambda + \varepsilon$. A fortiori, $\operatorname{dist}(\varphi; W) \leqslant \lambda + \varepsilon$. Since $\varepsilon > 0$ was arbitrary, $\operatorname{dist}(\varphi; W) \leqslant \lambda$.

DEFINITION 1.6. A family of functions $F \subset \mathscr{C}(X; E)$ is said to *vanish collectively at infinity* if, for each $\varepsilon > 0$, there is a compact subset $K \subset X$ such that $\|f(x)\| < \varepsilon$ for all $x \notin K$ and $f \in F$.

EXAMPLE 1.7. Let $F \subset \mathscr{C}_0(X; E)$ be a totally bounded subset. Then F vanishes collectively at infinity. Indeed, let $\varepsilon > 0$ be given. There exists a finite set $\{f_1, f_2, \ldots, f_n\} \subset F$ such that, for each $f \in F$, there is $1 \leqslant i \leqslant n$ with $\|f - f_i\| < \varepsilon/2$. For each $1 \leqslant i \leqslant n$, there is a compact subset $K_i \subset X$ such that $\|f_i(x)\| < \varepsilon/2$ for all $x \notin K_i$. Let K be the union $K_1 \cup K_2 \cup \cdots \cup K_n$. Then for all $x \notin K$ and $f \in F$, $\|f(x)\| < \varepsilon$.

PROPOSITION 1.8. *Let $F \subset \mathscr{C}(X; E)$ be a family which vanishes collectively at infinity and let $W \subset \mathscr{C}_0(X; E)$. The carrier $\varphi(x) = \{f(x); f \in F\}$, $x \in X$, vanishes at infinity with respect to W.*

Proof. If $F \subset \mathscr{C}(X; E)$ vanishes collectively at infinity and $w \in \mathscr{C}_0(X; E)$, then $G = \{f - w; f \in F\}$ vanishes collectively at infinity too. Let K be the compact set such that $\|f(x) - w(x)\| < \varepsilon$ for all $x \notin K$ and $f \in F$. Then $\varphi(x) \subset B(w(x); \varepsilon)$ for all $x \notin K$. Hence

$$X \backslash \{x \in X; \varphi(x) \subset B(w(x); \varepsilon)\} \subset K,$$

and so the set

$$\{x \in X; \varphi(x) \cap (E \backslash B(w(x); \varepsilon)) \neq \varnothing\}$$

is relatively compact.

THEOREM 1.9. *Let A and W be as in Theorem 1.5. Let $F \subset \mathscr{C}_0(X; E)$ be a totally bounded subset and define for $x \in X$, $\varphi(x) = \{f(x); f \in F\}$. Then*

$$\operatorname{dist}(\varphi; W) = \sup_{x \in X} \operatorname{dist}(\varphi | \Delta(x); W | \Delta(x)).$$

Proof. By Example 1.3, φ is upper semicontinuous, and by Example 1.7 and Proposition 1.8, φ vanishes at infinity with respect to any $W \subset \mathscr{C}_0(X; E)$. It remains to apply Theorem 1.5.

THEOREM 1.10. *Let X be a compact Hausdorff space; let $A \subset \mathscr{C}(X; \mathbb{K})$ be a self-adjoint subalgebra, and let $W \subset \mathscr{C}(X; E)$ be an A-module. Let $F \subset \mathscr{C}(X; E)$ be a bounded and equicontinuous subset and define $\varphi(x) = \{f(x); f \in F\}$ for all $x \in X$. Then*

$$\operatorname{dist}(\varphi; W) = \sup_{x \in X} \operatorname{dist}(\varphi | \Delta(x); W | \Delta(x)).$$

Proof. By Example 1.3, φ is upper semicontinuous. Since X is compact, any $F \subset \mathscr{C}(X; E)$ vanishes collectively at infinity, and then by Proposition 1.8, φ vanishes at infinity with respect to any $W \subset \mathscr{C}(X; E) = \mathscr{C}_0(X; E)$.

Let us apply Theorems 1.9 and 1.10 to the problem of best *simultaneous approximation* in $\mathscr{C}_0(X; E)$. For a normed space $(N, \|\cdot\|)$, and a non-empty subset $W \subset N$, one defines for each bounded subset F of N the *relative Chebyshev radius of F with respect to W*:

$$\operatorname{rad}_W(F) = \inf_{g \in W} \sup_{f \in F} \|f - g\|.$$

When $W = N$, one speaks of the *Chebyshev radius of* F and writes

$$\operatorname{rad}(F) = \inf_{g \in N} \sup_{f \in F} \|f - g\|.$$

Assume now $N = \mathscr{C}_0(X; E)$ with the sup-norm and let φ be the carrier from X into E defined by

$$\varphi(x) = \{f(x); f \in F\}$$

for all $x \in X$, where $F \subset \mathscr{C}_0(X; E)$ is some bounded subset. Then, for any $W \subset \mathscr{C}_0(X; E)$,

$$\operatorname{dist}(\varphi; W) = \operatorname{rad}_W(F).$$

Also, if Δ is any equivalence relation on X with closed equivalence classes, then

$$\operatorname{dist}(\varphi | \Delta(x);\ W | \Delta(x)) = \operatorname{rad}_{W|\Delta(x)}(F | \Delta(x))$$

for all $x \in X$. Hence the following result follows from Theorems 1.9 and 1.10.

THEOREM 1.11. *Let $A \subset \mathscr{C}_b(X; \mathbb{K})$ be a self-adjoint subalgebra, and let $W \subset \mathscr{C}_0(X; E)$ be an A-module. For any totally bounded subset $F \subset \mathscr{C}_0(X; E)$, or for any bounded and equicontinuous subset $F \subset \mathscr{C}(X; E)$, if X is compact, we have*

$$\operatorname{rad}_W(F) = \sup_{x \in X} \operatorname{rad}_{W|\Delta(x)}(F | \Delta(x)),$$

where Δ is the equivalence relation defined by A.

COROLLARY 1.12. *Let V be a closed subspace of E, and let F be as in Theorem* 1.11. *Then*

$$\operatorname{rad}_{\mathscr{C}_0(X; V)}(F) = \sup_{x \in X} \operatorname{rad}_V(F(x)).$$

Proof. The subspace $W = \mathscr{C}_0(X; V) = \{g \in \mathscr{C}_0(X; E);\ g(X) \subset V\}$ is a $\mathscr{C}_b(X; \mathbb{K})$-module and $W(x) = V$ for each $x \in X$. If one introduces the carrier $\varphi(x) = \{f(x);\ f \in F\} = F(x)$, then for each point $x \in X$ one has $\operatorname{dist}(\varphi(x); W(x)) = \operatorname{dist}(\varphi(x); V) = \operatorname{rad}_V(F(x))$, and the result follows from Theorem 1.11.

DEFINITION 1.13. Let B be a bounded subset of a normed space $(N, \|\cdot\|)$. The *Kuratowksi measure of non-precompactness* of B is the greatest lower bound $\alpha(B)$ of all $r > 0$ such that there is a finite set $J \subset B$ such that $B \subset \bigcup \{B(f; r); f \in J\}$.

Clearly, $\alpha(B) = 0$ if, and only if, B is totally bounded.

THEOREM 1.14. *Let A, Δ and W be as in Theorem* 1.11. *For any bounded subset $F \subset \mathscr{C}_0(X; E)$ one has*

$$r(F; W) \leqslant \text{rad}_W(F) \leqslant r(F; W) + \alpha(F),$$

where

$$r(F; W) = \sup_{x \in X} \text{rad}_{W|\Delta(x)} (F|\Delta(x)).$$

Proof. Clearly, $r(F; W) \leqslant \text{rad}_W(F)$. Let $r > 0$ be such that there is a finite set $J \subset F$ such that $F \subset \bigcup \{B(f; r); f \in J\}$. Fix $g \in W$. For each $f \in F$ there is some f_J in J such that $\|f - f_J\| < r$. Hence,

$$\sup_{f \in F} \| g - f \| \leqslant \sup_{f \in J} \| g - f \| + r,$$

and from this it follows that

$$\text{rad}_W(F) \leqslant \text{rad}_W(J) + r.$$

By Theorem 1.11, since J is finite,

$$\text{rad}_W(J) = r(J; W).$$

On the other hand, $J \subset F$ implies $r(J; W) \leqslant r(F; W)$. Hence

$$\text{rad}_W(F) \leqslant r(F; W) + r,$$

and so

$$\text{rad}_W(F) \leqslant r(F; W) + \alpha(F).$$

COROLLARY 1.15. *Let V be a closed subspace of E, and let $F \subset \mathscr{C}_0(X; E)$ be a bounded subset. Then*

$$r(F; V) \leqslant \text{rad}_{\mathscr{C}_0(X; V)}(F) \leqslant r(F; V) + \alpha(F),$$

where

$$r(F; V) = \sup_{x \in X} \text{rad}_V(F(x)).$$

Proof. Apply Theorem 1.14 to $W = \mathscr{C}_0(X; V) = \{ g \in \mathscr{C}_0(X; E); g(x) \subset V\}$ and $A = \mathscr{C}_b(X; \mathbb{K})$.

2. Weierstrass–Stone Subspaces

Let Δ be an equivalence relation of X, and φ a Δ-bounded carrier of X into E; that is,

$$\varphi(\Delta(x)) = \bigcup \{\varphi(t);\, t \in \Delta(x)\}$$

is a bounded subset of E, for each $x \in X$. Define

$$\delta(\varphi) = \sup_{x \in X} \operatorname{rad}(\varphi(\Delta(x)).$$

Theorem 2.1. *Let $A \subset \mathscr{C}_b(X; \mathbb{K})$ be a self-adjoint subalgebra and let Δ be the equivalence relation defined by A. Let $W \subset \mathscr{C}_0(X; E)$ be an A-module such that, for each $x \in X$ and $z \in E$, there is some $g \in W$ such that $g(t) = z$ for all $t \in \Delta(x)$. Then for any Δ-bounded carrier φ from X into E which is upper semicontinuous and vanishes at infinity with respect to W, we have*

$$\operatorname{dist}(\varphi; W) \leqslant \delta(\varphi).$$

Proof. By Theorem 1.5 we have

$$\operatorname{dist}(\varphi; W) = \sup_{x \in X} \inf_{g \in W} \sup_{t \in \Delta(x)} \sup_{y \in \varphi(t)} \|y - g(t)\|.$$

Let $x \in X$. For each $z \in E$, choose $g_z \in W$ such that $g_z(t) = z$ for all $t \in \Delta(x)$. Then

$$\inf_{g \in W} \sup_{t \in \Delta(x)} \sup_{y \in \varphi(t)} \|y - g(t)\|$$
$$\leqslant \sup_{t \in \Delta(x)} \sup_{y \in \varphi(t)} \|y - g_z(t)\| = \sup_{y \in \varphi(\Delta(x))} \|y - z\|.$$

Since $z \in E$ was arbitrary, we have

$$\inf_{g \in W} \sup_{t \in \Delta(x)} \sup_{y \in \varphi(x)} \|y - g(t)\|$$
$$\leqslant \inf_{z \in E} \sup_{y \in \varphi(\Delta(x))} \|y - z\|,$$

and from this it clearly follows that $\operatorname{dist}(\varphi; W) \leqslant \delta(\varphi)$.

Now let $\pi: X \to Y$ be a proper continuous surjection, where Y is another locally compact Hausdorff space. Since π is proper, $\pi^{-1}(K)$ is compact in X for each compact set $K \subset Y$, and then π^* maps $\mathscr{C}_0(Y; E)$ into $\mathscr{C}_0(X; E)$, where π^* is the map $h \to h \circ \pi$. The sets $\pi^{-1}(y)$, for $y \in Y$, are the equivalence classes of the equivalence relation defined on X by the subalgebra $A_\pi \subset \mathscr{C}_b(X; \mathbb{R})$ of all $g \circ \pi$ with $g \in \mathscr{C}_b(Y; \mathbb{R})$.

Let $f \in \mathscr{C}_0(X; E)$ be given. Since π is proper, $\pi^{-1}(y)$ is compact, and then $f(\pi^{-1}(y))$ is compact, hence bounded in E, for each $y \in Y$. Let us define

$$\delta(f) = \sup\{\text{rad}(f(\pi^{-1}(y))); y \in Y\}.$$

Let $W \subset \mathscr{C}_0(X; E)$ be a subset such that each $g \in W$ is constant on every $\pi^{-1}(y)$, $y \in Y$. Then

$$\|f - g\| = \sup_{y \in Y} \sup_{t \in \pi^{-1}(y)} \|f(t) - g(t)\| \geqslant \delta(f)$$

for all $g \in W$. Hence

$$\delta(f) \leqslant \text{dist}(f; W)$$

for all $W \subset \mathscr{C}_0(X; E)$ such that each $g \in W$ is constant on the sets $\pi^{-1}(y)$, $y \in Y$.

THEOREM 2.2. *Let Y be a locally compact Hausdorff space and let M be a $\mathscr{C}_b(Y; \mathbb{K})$-submodule of $\mathscr{C}_0(Y; E)$ such that $M(y) = E$ for all $y \in Y$. Then for any $f \in \mathscr{C}_0(X; E)$, we have*

$$\text{dist}(f; \pi^*(M)) = \delta(f)$$

if $\pi: X \to Y$ is a proper continuous surjection.

Proof. Clearly, $W = \pi^*(M) \subset \mathscr{C}_0(X; E)$ is an A_π-module and by Theorem 2.1

$$\text{dist}(f; \pi^*(M)) \leqslant \delta(f).$$

Since $g \in \pi^*(M)$ is constant on the sets $\pi^{-1}(y)$, $y \in Y$, then by the remarks made before we have $\delta(f) \leqslant \text{dist}(f; \pi^*(M))$.

Remark 1.3. In Olech [4] the formula $\text{dist}(f; \pi^*(\mathscr{C}_0(Y; E))) = \delta(f)$ was proved for X and Y compact, and E a uniformly convex Banach space (see [4, Theorem 2]). Indeed, Olech has the formula as a corollary to his more difficult result that, under the hypothesis above, each $f \in \mathscr{C}_0(X; E)$ has a best approximation g from $W = \pi^*(\mathscr{C}_0(Y; E))$, i.e., $\text{dist}(f; W) = \|f - g\|$, and for such g, $\|f - g\| = \delta(f)$. We have seen that the formula for the distance is true in general and does not depend on the existence of a best approximation from W, indeed it is a corollary of the "strong" version of the Weierstrass–Stone theorem that we proved, namely, Theorem 1.5.

Let us now consider the case of best simultaneous approximation. Consider then a totally bounded subset $F \subset \mathscr{C}_0(X; E)$ and the associated carrier φ from X into E defined by $\varphi(x) = \{f(x); f \in F\}$ for all $x \in X$. Since

F is bounded, it follows that φ is a Δ-bounded carrier for any equivalence ralation Δ on X.

For each $y \in Y$ define

$$F(\pi^{-1}(y)) = \bigcup \{f(\pi^{-1}(y)); f \in F\}$$

and

$$\delta(F) = \sup\{\mathrm{rad}(F(\pi^{-1}(y))); y \in Y\}.$$

Then $\delta(F) = \delta(\varphi)$, and by Theorem 2.1

$$\mathrm{rad}_W(F) \leqslant \delta(F)$$

for $W = \pi^*(M)$, where M is as in Theorem 2.2. Conversely, each $g \in \pi^*(M)$ is constant on $\pi^{-1}(y)$ for every $y \in Y$. Thus

$$\begin{aligned}
\mathrm{dist}(\varphi; g) &= \sup_{y \in Y} \sup_{t \in \pi^{-1}(y)} \sup_{z \in \varphi(t)} \|z - g(t)\| \\
&\geqslant \sup_{y \in Y} \inf_{v \in E} \sup_{t \in \pi^{-1}(y)} \sup_{z \in \varphi(t)} \|z - v\| \\
&= \sup_{y \in Y} \inf_{v \in E} \sup_{t \in \pi^{-1}(y)} \sup_{f \in F} \|f(t) - v\| \\
&= \sup_{y \in Y} \mathrm{rad}(F(\pi^{-1}(y))) = \delta(F).
\end{aligned}$$

Hence

$$\delta(F) \leqslant \mathrm{dist}(\varphi; W) = \mathrm{rad}_W(F).$$

We have thus proved the following:

THEOREM 2.4. *Let Y be a locally compact Hausdorff space and let M be a $\mathscr{C}_b(Y; \mathbb{K})$-submodule of $\mathscr{C}_0(Y; E)$ such that $M(y) = E$ for all $y \in Y$. Then, for any totally bounded subset $F \subset \mathscr{C}_0(X; E)$, we have*

$$\mathrm{rad}_{\pi^*(M)}(F) = \sup\{\mathrm{rad}(F(\pi^{-1}(y))); y \in Y\}$$

if $\pi: X \to Y$ is a proper continuous surjection.

COROLLARY 2.5. *Let Y, M and π be as in Theorem 2.4. For any bounded subset $F \in \mathscr{C}_0(X; E)$ we have*

$$\delta(F) \leqslant \mathrm{rad}_{\pi^*(M)}(F) \leqslant \delta(F) \leqslant \delta(F) + \alpha(F)$$

where $\delta(F) = \sup\{\mathrm{rad}(F(\pi^{-1}(y))); y \in Y\}$.

Proof. The inequality $\delta(F) \leqslant \text{rad}_{\pi^*(M)}(F)$ was established in the proof of Theorem 2.4. Indeed, in that part of the proof the arguments depend only on F being bounded. The inequality $\text{rad}_{\pi^*(M)}(F) \leqslant \delta(F) + \alpha(F)$ follows from Theorem 2.4 in the same way as Theorem 1.14 follows from Theorem 1.11.

3. Weighted Approximation of Set–Valued Mappings

Throughout this section X stands for a completely regular Hausdorff space and E stands for a locally convex Hausdorff space over $\mathbb{K}$, where $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$. If V is a family of upper semicontinuous non-negative functions on X, we denote by $\mathscr{C}V_\infty(X; E)$ the vector subspace of $\mathscr{C}(X; E)$ consisting of all $f \in \mathscr{C}(X; E)$ such that vf vanishes at infinity, for each $v \in V$, that is, $\{x \in X; v(x)\,p(f(x)) \geqslant \varepsilon\}$ is compact, for each $\varepsilon > 0$, and each continuous seminorm p on E. We introduce a locally convex topology on $\mathscr{C}V_\infty(X; E)$ by considering the family of seminorms $f \to \sup\{v(x)\,p(f(x));\ x \in X\}$, where $v \in V$ and p is a continuous seminorm on E. Without loss of generality, we may assume that V is upper directed in the following sense: given v and u in V, there are $t > 0$ and $w \in V$ such that

$$\max(v(x),\ u(x)) \leqslant tw(x)$$

for all $x \in X$.

Let us consider a subalgebra $A \subset \mathscr{C}(X; \mathbb{K})$ and a vector subspace $W \subset \mathscr{C}V_\infty(X; E)$ which is an A-module, that is, $AW \subset W$ under pointwise multiplication operation. The *weighted approximation problem* posed by Nachbin [3, p. 289] consists in asking for a description of the closure of W in $\mathscr{C}V_\infty(X; E)$. This was solved by Nachbin in the case $\mathbb{K} = \mathbb{R}$, or $\mathbb{K} = \mathbb{C}$ and A self-adjoint, under the following hypothesis: every $a \in A$ is bounded on the support of every $v \in V$. (See [3, Theorem 1, p. 295].)

Definition 3.1. Let φ be a carrier of X into E. We say that φ is *lower semicontinuous* (l.s.c.) *with respect to* $W \subset \mathscr{C}V_\infty(X; E)$ if for each $w \in W$, $v \in V$, $\varepsilon > 0$ and p a continuous seminorm on E, the set

$$\{x \in X; \varphi(x) \cap B_{v(x)p}(w(x); \varepsilon) \neq \varnothing\}$$

is open, where

$$B_{v(x)p}(w(x); \varepsilon) = \{t \in E; v(x)\,p(t - w(x)) < \varepsilon\}.$$

Similarly, we say that φ *vanishes at infinity with respect to* $W \subset \mathscr{C}V_\infty(X; E)$ if, for each w, v, ε and p as above, the set

$$\{x \in X; \varphi(x) \cap (E \backslash B_{v(x)p}(w(x); \varepsilon)) \neq \varnothing\}$$

is relatively compact.

EXAMPLE 3.2. *If* $f \in \mathscr{C}V_\infty(X; E)$, then $\varphi(x) = \{f(x)\}$, $x \in X$, is lower semicontinuous and vanishes at infinity with respect to any $W \subset \mathscr{C}V_\infty(X; E)$.

DEFINITION 3.3. Let φ be a carrier of X into E, and let $W \subset \mathscr{C}V_\infty(X; E)$ be a non-empty subset. We say that φ is *W-admissible* if φ is lower semicontinuous and vanishes at infinity with respect to W.

DEFINITION 3.4. Let φ be a carrier of X into E, and let $W \subset \mathscr{C}V_\infty(X; E)$ be a non-empty subset. Given $v \in V$, $\varepsilon > 0$ and p a continuous seminorm on E, we say that $g \in W$ is a (v, ε, p) – *approximate W-selection for* φ, if $g(x) \in \varphi(x) + B_{v(x)p}(0; \varepsilon)$ for all $x \in X$. And we say that φ can be *V-approximated by elements of W* if φ has (v, ε, p)-approximate W-selections for all choices of $v \in V$, $\varepsilon > 0$ and p a continuous seminorm on E.

The *weighted approximation problem for set-valued mappings* consists in, given a subspace $W \subset \mathscr{C}V_\infty(X; E)$, find necessary and sufficient conditions for a W-admissible carrier φ to be V-approximated by elements of W.

If $f \in \mathscr{C}V_\infty(x; E)$ and for all $x \in X$, $\varphi(x) = \{f(x)\}$, then clearly φ can be V-approximated by elements of $W \subset \mathscr{C}V_\infty(X; E)$ if, and only if, f belongs to the closure of W in $\mathscr{C}V_\infty(X; E)$.

Let us remark that if φ is a carrier and W is an A-module, where $A \in \mathscr{C}_b(X; \mathbb{K})$, then in order to prove that φ can be V-approximated by elements of W, we may assume without loss of generality that A is *closed* in $\mathscr{C}_b(X; \mathbb{K})$ in the sup-norm. Indeed, let B denote the closure of A and assume that φ can be V-approximated by elements of the B-module W' generated by W. Let $v \in V$, $\varepsilon > 0$ and p a continuous seminorm on E be given. Then we can find $h \in W'$ such that $h(x) \in \varphi(x) + B_{v(x)p}(0; \varepsilon/2)$ for all $x \in X$. Suppose $h = \sum_{i=1}^n b_i w_i$ with $b_i \in B$, $w_i \in W$, $i = 1, 2, \ldots, n$. Choose $\delta > 0$ so small that $\delta \sum_{i=1}^n v(x) p(w_i(x)) < \varepsilon/2$ for all $x \in X$. Let $a_i \in A$ be such that $|a_i(x) - b_i(x)| < \delta$ for all $x \in X$, $i = 1, 2, \ldots, n$. Then $g = \sum_{i=1}^n a_i w_i$ belongs to the A-module W and $g(x) \in \varphi(x) + B_{v(x)p}(0; \varepsilon)$ for all $x \in X$.

THEOREM 3.5. *Let* $A \subset \mathscr{C}_b(X; \mathbb{K})$ *be a self-adjoint subalgebra and let* $W \subset \mathscr{C}V_\infty(X; E)$ *be an A-module. Let* φ *be a W-admissible carrier of X into E such that* $\varphi(x)$ *is convex for each* $x \in X$. *We can V-approximate* φ *by elements of W if, and only if, for each* $x \in X$, $\varphi | \Delta(x)$ *can be* $V | \Delta(x)$-*approximated by elements of* $W | \Delta(x)$, *where* Δ *is the equivalence relation defined by A.*

Proof. The condition is obviously necessary. Conversely, assume that φ is such that, for each $x \in X$, $\varphi | \Delta(x)$ can be $V | \Delta(x)$-approximated by elements of $W | \Delta(x)$.

Let $v \in V$, $\varepsilon > 0$ and p a continuous seminorms on E be given.

Without loss of generality we may assume that A contains the constants. Indeed, the algebra B generated by A and the constants defines the same equivalence relation on X as A, and W is a B-module. By the remarks preceding the theorem we may also assume that A is a closed subalgebra of $\mathscr{C}_b(X; \mathbb{K})$.

For every $x \in X$, there is some $g_x \in W$ such that for all $t \in \Delta(x)$, $g_x(t) \in \varphi(t) + B_{v(x)p}(0; \varepsilon)$. Let $\psi(t)$ be the set $\varphi(t) + B_{v(t)p}(0; \varepsilon)$, for all $t \in X$. Since φ is lower semicontinuous with respect to W, the set

$$U_x = \{x' \in X; g_x(x') \in \psi(x')\}$$

is open and, by hypothesis, $U_x \supset \Delta(x)$. On the other hand, since φ vanishes at infinity with respect to W, the set

$$S_x = \{x' \in X; \varphi(x') \cap (E \backslash B_{v(x)p}(g_x(x'); \varepsilon) \neq \varnothing\}$$

is relatively compact. Now $K_x = X \backslash U_x$ is closed and contained in S_x; therefore K_x is compact. By [3, Lemma 1], there exists a finite set $\{x_1, x_2, ..., x_n\} \subset X$ and, for each $1 \leqslant i \leqslant n$, there is $\varphi_i \in A$ such that $\varphi_i \geqslant 0$, $\varphi_i | K_{x_i} = 0$ and $\sum_{i=1}^n \varphi_i = 1$ on X. Then $g = \sum_{i=1}^n \varphi_i g_{x_i}$ belongs to W and $g(x) \in \varphi(x) + B_{v(x)p}(0; \varepsilon)$, for all $x \in X$.

In fact, either $x \in K_{x_i}$ and then $\varphi_i(x) = 0$; or else $x \notin K_{x_i}$ and then $g_{x_i}(x) \in \varphi(x) + B_{v(x)p}(0; \varepsilon)$. Hence $g(x)$ is a convex combination of elements which lie in the convex set $\varphi(x) + B_{v(x)p}(0; \varepsilon)$. Therefore $g(x) \in \varphi(x) + B_{v(x)p}(0; \varepsilon)$ for all $x \in X$; i.e., g is a (v, ε, p)-approximate W-selection for φ.

Remark 3.6. Notice that we proved the following stronger result: if for each $x \in X$, $\varphi | \Delta(x)$ has a (v, ε, p)-approximate $(W | \Delta(x))$-selection, then φ has a (v, ε, p)-approximate W-selection.

Hence we have shown that in the approximation lemma of Blatter [1, p. 37], neither paracompactness of T nor compactness of the equivalence classes of the equivalence relation R are needed, if instead of the usual notion of lower semicontinuity for set-valued mappings, one requires l.s.c. in the sense of Definition 3.1. Notice that, when a carrier φ from X into E is l.s.c. with respect to the set $W \subset \mathscr{C}V_\infty(X; E)$ of all *constant* functions, then $\{x \in X; \varphi(x) \cap U \neq \varnothing\}$ is open in X, for every open set U of X; that is, φ is l.s.c. in the usual sense. One example of V such that $\mathscr{C}V_\infty(X; E)$ actually contains the constant functions in given by the set V of all characteristic functions of compact subsets. Indeed, in this case $\mathscr{C}V_\infty(X; E)$ is just $\mathscr{C}(X; E)$ with the compact-open topology.

Another example is given by the set V of all positive functions belonging to $\mathscr{C}_0(X; \mathbb{R})$ for a locally compact space X. In this case $\mathscr{C}V_\infty(X; E)$ is $\mathscr{C}_b(X; E)$ with the strict topology.

However, when V is the set of all positive constant functions on X, $\mathscr{C}V_\infty(X; E)$ is $\mathscr{C}_0(X; E)$ with the uniform topology and, unless X is compact, it contains no non-zero constant function.

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